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ON THE LIBRATION BOUNDARIES OF A SATELLITE IN CIRCULAR ORBIT UNDER THE ACTION OF POTENTIAL PERTURBING FORCES

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The parameters of the final rotation of a satellite with respect to its position of stable equilibrium are chosen as variables convenient for estimating the potential energy of the perturbing forces. It is shown that the perturbing forces and deviations of the satellite satisfy inequalities (3.4) and (3.6). These inequalities constitute the conditions of (λ, A, t_0, T)- stability [1] of the satellite's equilibrium.

1. Let us assume that the center of mass of a satellite moves as a material point along a Keplerian circular orbit and let us introduce the right-handed rectangular coordinate systems $Ox_1x_2x_3$ and $Oy_1y_2y_3$. We direct the axes of the first of these systems along the principal central axes of inertia of the satellite. The second system is the orbital system (y_1 lies along the velocity, y_2 along the normal to the orbital plane, y_3 along the radius vector).

The potential energy of the gravitational and inertial forces acting on the satellite is given by the expression [3]

$$W = a\alpha_{21}^2 + b\alpha_{23}^2 + c\alpha_{31}^2 + d\alpha_{32}^2, \quad \alpha_{ij} = \cos y_i x_j \tag{1.1}$$

$$a = {}^1/_2\omega^2 (A_2 - A_1), \; b = {}^1/_2 \omega^2(A_2 - A_3), \; c = {}^3/_2 \omega^2 (A_1 - A_3), \; d = {}^3/_2 \omega^2 (A_2 - A_3)$$

in the orbital coordinate system. Here ω is the Keplerian orbital angular velocity and A_i are the principal central moments of inertia of the satellite. The coefficients a, b, c, d are related to each other by the self-evident equations

$$d = 3b = c + 3a \tag{1.2}$$

The relative motions of the satellite in the orbital coordinate system have the energy integral H,

$$H = T + W = h, \quad 2T = A_1p_1^2 + A_2p_2^2 + A_3p_3^2 \tag{1.3}$$

Here T is the kinetic energy of the relative motions and p_i are the projections of the relative angular velocity of the satellite onto the axes x_i.

The table of cosines α_{ij} expressed in terms of the Rodrigues-Hamilton parameters λ_0, λ_i $(i = 1, 2, 3)$ can be written out in the following form:

	x_1	x_2	x_3
y_1	$\lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2$	$2(-\lambda_0\lambda_3 + \lambda_1\lambda_2)$	$2(\lambda_0\lambda_2 + \lambda_1\lambda_3)$
y_2	$2(\lambda_0\lambda_3 + \lambda_1\lambda_2)$	$\lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2$	$2(-\lambda_0\lambda_1 + \lambda_2\lambda_3)$
y_3	$2(-\lambda_0\lambda_2 + \lambda_1\lambda_3)$	$2(\lambda_0\lambda_1 + \lambda_2\lambda_3)$	$\lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2$

$$\lambda_0 = \cos {}^1/_2\,\chi, \qquad \lambda_i = \gamma_i \sin {}^1/_2\,\chi \qquad (i = 1, 2, 3), \qquad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$$

Here γ_i are the cosines of the angles between the final-rotation axis and the coordinate axes (which are equal in the system $Ox_1x_2x_3$ and $Oy_1y_2y_3$) ; χ is the final-rotation angle. In these variables we have

$$W = \sin^2\chi\, F, \qquad F = [(b + d)\gamma_2^2\gamma_3^2 + c\gamma_3^2\gamma_1^2 + a\gamma_1^2\gamma_2^2]\,\mathrm{tg}^2\,{}^1/_2\,\chi + \\ + 2(a - b - c + d)\gamma_1\gamma_2\gamma_3\,\mathrm{tg}\,{}^1/_2\chi + (b + d)\gamma_1^2 + c\gamma_2^2 + a\gamma_3^2 \qquad (1.4)$$

Let us assume that $A_2 > A_1 > A_3$. Then $a, b, c, d > 0$, and the satellite is in a stable equilibrium position for $x_i = y_i$.

2. Let us determine the minimum and maximum with respect to γ_i ($\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$) of the function W for a fixed value of $\chi < {}^1/_2\,\pi$

$$\min W = \sin^2\chi \min F, \qquad \max W = \sin^2\chi \max F \qquad (2.1)$$

We propose to show that

$$\min F = \min\{a, c\}, \qquad \max F = b + d \qquad (2.2)$$

From (1.2) we obtain the inequality

$$b + d - a - c > \qquad (2.3)$$

Considering the values of the function F for

$$\gamma_1 = \pm 1, \quad \gamma_2 = \gamma_3 = 0 \quad (123) \qquad (2.4)$$

we find from (1.4) and (2.3) that

$$\min F \leqslant \min\{a, c\} < b + d \leqslant \max F \qquad (2.5)$$

Let us consider the three cases

$$a - b - c + d = 0, \quad a - b - c + d > 0, \quad a - b - c + d < 0 \qquad (2.6)$$

In the case (2.6.1) the first relation of (2.2) is self-evident, and (2.2.2) is a consequence of (2.5) and of the following relations valid under conditions (2.3) and (2.6.1):

$$F = b + d - (b + d - \lambda a - c)\gamma_2^2 - (b + d - a - \lambda c)\gamma_3^2 + \lambda(b + d - a - c)\gamma_2^2\gamma_3^2 - \\ - \lambda(a\gamma_2^4 + c\gamma_3^4) \leqslant b + d - (b + d - a - c)(\gamma_2^2 + \gamma_3^2 - \gamma_2^2\gamma_3^2) \leqslant b + d \\ (\lambda = \mathrm{tg}^2\,{}^1/_2\,\chi,\ 0 < \lambda < 1)$$

To prove (2.2) in the two remaining cases (2.6.2) and (2.6.3) we must consider the functions W^-, F^-, W^+, F^+ which differ from W and F by the coefficients a^-, b^-, c^-, d^- and a^+, b^+, c^+, d^+, respectively, where

$$0 < a^- \leqslant a \leqslant a^+, \quad 0 < b^- \leqslant b \leqslant b^+, \quad 0 < c^- \leqslant c \leqslant c^+, \quad 0 < d^- \leqslant d \leqslant d^+ \qquad (2.7) \\ a^\pm - b^\pm - c^\pm + d^\pm = 0, \qquad b^\pm + d^\pm - a^\pm - c^\pm > 0$$

Then, first, expressions (1.1), (1.4.1), (2.1) yield

$$W^- \leqslant W \leqslant W^+, \quad F^- \leqslant F \leqslant F^+, \quad \min F \geqslant \min F^-, \quad \max F \leqslant \max F^+ \qquad (2.8)$$

and second, in accordance with case (2.6.1),

$$\min F^\pm = \min\{a^\pm, c^\pm\}, \qquad \max F^\pm = b^\pm + d^\pm \qquad (2.9)$$

On the basis of relations (1.2) we can readily verify that conditions (2.7) are satisfied by the following values:

in the case (2.6.2), (2.10)

$$a^- = a,\ b^- = b,\ c^- = c,\ d^- = -a + b + c,\ a^+ = a,\ b^+ = b,\ c^+ = a - b + d,\ d^+ = d$$

in the case (2.6.3), (2.11)

$$a^- = a,\ b^- = a - c + d,\ c^- = c,\ d^- = d,\ a^+ = b + c - d,\ b^+ = b,\ c^+ = c,\ d^+ = d$$

Comparing (2.5) with (2.8.3), (2.8.4) after substituting (2.9) with values (2.10), (2.11) into them, we see that (2.2) are also valid in cases (2.6.2), (2.6.3).

Thus, (2.2) has been proved completely.

We note that $\min F$ and $\max F$ do not depend on χ. Hence, (2.1) implies that the minimum and maximum of χ under the conditions

$$W = h, \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$$

can be determined from the relations

$$\min \sin^2 \chi = h / (b + d), \quad \max \sin^2 \chi = \sin^2 \chi^*(h) = h / \min \{a, c\} \qquad (2.12)$$

and are attained for values (2.4).

Let us rewrite (2.12.2), (2.1) with allowance for (1.1.2), (2.2.2) as

$$\begin{gathered} \min W = {}^1/_2\, \omega^2 \sin^2 \chi \min \{(A_2 - A_1),\ 3\,(A_1 - A_3)\} \\ \max W = 2\omega^2\,(A_2 - A_3) \sin^2 \chi \\ \max \sin^2 \chi = \sin^2 \chi^*\,(h) = 2h\omega^{-1} / \min \{(A_2 - A_1),\ 3\,(A_1 - A_3)\} \end{gathered} \qquad (2.13)$$

3. We infer from (1.3), (2.13.3) that if the energy of the satellite does not exceed h, then the angle of its deviation from the equilibrium position $x_i = y_i$ does not exceed $\chi^*(h)$. This result agrees with the estimates of the libration domains obtained directly in terms of the direction cosines in [2].

In other words, the equilibrium position $x_i = y_i$ of the satellite is $(\lambda, A, t_0, \infty)$-stable [1] if we take

$$H \leqslant \lambda, \quad \chi \leqslant A \qquad (A \geqslant \chi^*(\lambda))$$

as our domains (λ) and (A).

Let us suppose that the satellite is acted on by perturbing potential moments with the potential energy w. Under appropriate assumptions those moments can include the gyrostatic moments of masses rotating inside the satellite, the aerodynamic moments, the reactive moments due to the reactions of the air jets escaping from the satelite as a result of imperfect pressurization, the moments due to nonsphericity of the gravitational field in the case of an equatorial satellite, and other moments.

If the sum of all such perturbing moments is not larger than some positive number M in absolute value, then, clearly,

$$|w| \leqslant \chi M \qquad (3.1)$$

The equilibrium is $(\lambda, A, t_0, \infty)$-stable if [1]

$$\max_{(\lambda)} V < \min_{[A]} V, \quad V = H + w = \text{const} \qquad (3.2)$$

where $[A]$ is the boundary of the domain (A).

If as our domains (λ) and (A) we take

$$H \leqslant \lambda, \quad H \leqslant A \qquad (3.3)$$

then, by virtue of (2.13.3) and (3.1),

$$\max_{(\lambda)} V \leqslant \lambda + M\chi^*(\lambda), \quad \min_{[A]} V \geqslant A - M\chi^*(A)$$

and inequality (3.2) is fulfilled for

$$(A - \lambda) / (\chi^*(A) + \chi^*(\lambda)) \geqslant M \qquad (3.4)$$

If the domains (λ) and (A) are defined in some other way, then the numbers λ° and A° must be substituted for λ and A in (3.4),

$$\lambda^\circ = \max_{(\lambda)} H, \quad A^\circ = \min_{[A]} H$$

Let us consider the practically interesting case where the domains (λ) and (A) are defined by the inequalities

$$p_1^2 + p_2^2 + p_3^2 < \omega^2\lambda^{(1)}, \ \chi < \lambda^{(2)}; \ p_1^2 + p_2^2 + p_3^2 < \omega^2 A^{(1)}, \ \chi < A^{(2)} \quad (3.5)$$

In this case we infer from (2.13) and (3.1) that

$$\max_{(\lambda)} V < {}^1/_2\, \omega^2 A_2\, \lambda^{(1)} + 2\omega^2 (A_2 - A_3) \sin^2 \lambda^{(2)} + M\lambda^{(2)}$$

$$\min_{[A]} V \geqslant {}^1/_2 \omega^2 \min \{A_3 A^{(1)},\ (A_2 - A_1) \sin^2 A^{(2)},\ 3 (A_1 - A_3) \sin^2 A^{(2)}\} - MA^{(2)}$$

and condition (3.2) becomes

$$ {}^1/_2\, \omega^2 \min \{A_3 A^{(1)},\ (A_2 - A_1) \sin^2 A^{(2)},\ 3 (A_1 - A_3) \sin^2 A^{(2)}\} - {}^1/_2\, \omega^2 A_2 \lambda^{(1)} - $$
$$ - 2\omega^2 (A_2 - A_3) \sin^2 \lambda^{(2)} > (A^{(2)} + \lambda^{(2)}) M \quad (3.6)$$

For comparison let us consider the case of nonpotential perturbing moments not exceeding M in absolute value. For example, let these be perturbations due to ellipticity of the orbit. Then, by virtue of the self-evident relation

$$dH / dt \leqslant M (p_1^2 + p_2^2 + p_3^2)^{1/2} \leqslant M (2H / A_3)^{1/2}$$

the equilibrium position under consideration is (λ, A, t_0, T)-stable with domains (λ) and (A) of the form (3.3), (3.5), respectively, under the conditions

$$M (T - t_0) \leqslant \sqrt{2A_3} (\sqrt{A} - \sqrt{\lambda})$$
$$\sqrt{A_3}\omega \, [(\min \{A_3 A^{(1)}, (A_2 - A_1) \sin^2 A^{(2)}, 3 (A_1 - A_3) \sin^2 A^{(2)}\})^{1/2} -$$
$$- (A_2\lambda^{(1)} + 4 (A_2 - A_3) \sin^2 \lambda^{(2)})^{1/2}] \geqslant M (T - t_0) \quad (3.7)$$

For example, let $A_1 = 4\times 10^{10}$ г·см², $A_2 = 6\times 10^{10}$ г·см², $A_3 = 3\times 10^{10}$ г·см², $\omega = 0.001\ \mathrm{sec}^{-1}$, $\lambda^{(1)} = \lambda^{(2)} = 0$, $A^{(1)} = \infty$, $A^{(2)} = 0.1$. Then the equilibrium position is $(\lambda, A, t_0, \infty)$- stable by virtue of (3.6) if the potential perturbing moments do not exceed 1000 g·cm²·sec⁻² in absolute value. For this value of the nonpotential perturbing moments condition (3.7.2) guarantees stability in the interval $T - t_0 < 2450$ sec only. During this time the center of mass of the satellite traveses less than one-half of its orbit.

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